Mathematical Analysis of Truncated Hexahedron (Cube)

Application of HCR's Theory of Polygon & HCR's Formula

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Introduction: A truncated hexahedron (cube) is a solid which has 8 congruent equilateral triangular & 6 congruent regular octagonal faces each having equal edge length. It is obtained by truncating a regular hexahedron (having 6 congruent faces each as a square) at the vertices to generate 8 equilateral triangular & 6 regular octagonal faces of equal edge length. For calculating all the parameters of a truncated hexahedron, we would use the equations of right pyramid & regular hexahedron (cube). When a regular hexahedron is truncated at the vertex, a right pyramid, with base as an equilateral triangle & certain normal height, is obtained. Since, a regular hexahedron has 8 vertices hence we obtain 8 truncated off congruent right pyramids each with an equilateral triangular base.

Truncation of a regular hexahedron (cube): For ease of calculations, let there be a regular hexahedron (cube) with edge length $PQ$ (unknown) & its centre at the point C. Now it is truncated at all 8 vertices to obtain a truncated hexahedron. Thus each of the congruent square faces with edge length $PQ$ is changed into a regular octagonal face with edge length $a$ (known) (see figure 2) & we obtain 8 truncated off congruent right pyramids with base as an equilateral triangle corresponding to 8 vertices of the parent solid. (See figure 1 which shows the truncation of a regular hexahedron (cube) & a right pyramid with equilateral triangular base of side $a$ & normal height $h$ being truncated from the regular hexahedron).

No. of congruent equilateral triangular faces in the truncated hexahedron

= no. of vertices in parent hexahedron = 8

No. of congruent regular octagonal faces in the truncated hexahedron

= no. of square faces in parent hexahedron = 6

No. of vertices in the truncated hexahedron = 8 x 3 = 24

Figure 1: A right pyramid with base as an equilateral triangle with side length $a$ & normal height $h$ is truncated off from argon™ proposed by Mr H.C. Rajpoot (year-2014) regular hexahedron (cube) with edge length $a$.
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Calculation of edge length of parent regular hexahedron:

Let \( a \) be the edge length of each face of a truncated hexahedron to be generated by truncating a parent regular hexahedron with edge length \( PQ \) (unknown).

\[
\angle APM = \frac{\angle APB}{2} \quad \text{(from figure 2)}
\]

\[
\frac{90^\circ}{2} = 45^\circ
\]

In right \( \Delta PMA \)

\[
\sin \angle APM = \frac{AM}{AP} \quad \text{or} \quad \sin 45^\circ = \frac{b}{AP}
\]

\[
\text{or} \quad AP = \frac{a}{2 \sin 36^\circ} = \frac{a}{2 \times \frac{1}{\sqrt{2}}} = \frac{a}{\sqrt{2}}
\]

\[
\therefore \text{edge length of parent hexahedron, } PQ = PA + AJ + JQ
\]

\[
= 2PA + AJ \quad \text{(since, } \ AB = AJ = JK = \ldots \ldots = \text{edge length of truncated hexahedron} = a)\n\]

\[
= 2 \times \frac{a}{\sqrt{2}} + a = a \sqrt{2} + a = (1 + \sqrt{2})a
\]

\[
\therefore \text{edge length of parent regular hexahedron, } PQ = (1 + \sqrt{2})a
\]

Above result shows that if we are to generate a truncated hexahedron with edge length \( a \) then we have to truncate all 8 vertices of a regular hexahedron (cube) of edge length \( (1 + \sqrt{2})a \).

Analysis of Truncated hexahedron by using equations of right pyramid & regular hexahedron

Now consider any of 8 truncated off congruent right pyramids having base as an equilateral triangle ABD with side length \( a \), normal height \( h \) & right angle \( 90^\circ \) between any two consecutive lateral edges (see figure 3 below)

Normal height \( (h) \) of truncated off right pyramid: We know that the normal height of any right pyramid with regular polygonal base is given as

\[
H = \frac{a}{2} \sqrt{\cot^2 \frac{\pi}{2} - \cot^2 \frac{n}{n}}
\]

\[
\therefore h = \frac{a}{2} \sqrt{\cot^2 \frac{90^\circ}{2} - \cot^2 \frac{\pi}{3}} = \frac{a}{2} \sqrt{(1)^2 - \left(\frac{1}{\sqrt{3}}\right)^2} \quad \text{(for equilateral triangular base, } n = 3)\]
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\[
= \frac{a}{2\sqrt{3}} = \frac{a}{\sqrt{6}}
\]

\[
h = \frac{a}{\sqrt{6}} \quad \ldots \ldots \ldots \ldots \ldots (\Omega)
\]

Figure 3: Normal distance \(H_T\) of equilateral triangular faces is always greater than the normal distance \(H_o\) of regular octagonal faces measured from the centre C of any truncated hexahedron.

Volume \(V'\) of truncated off right pyramid: We know that the volume of a right pyramid is given as

\[
\text{Volume} = \frac{1}{3} \left( \text{area of base (equilateral triangle)} \right) \times (\text{normal height})
\]

\[
\therefore V' = \frac{1}{3} \left( \frac{1}{4} \times a^2 \cdot \cot \frac{\pi}{n} \right) \times h = \frac{1}{3} \left( \frac{1}{4} \times 3 \times a^2 \cdot \cot \frac{\pi}{3} \right) \times \frac{a}{\sqrt{6}}
\]

\[
= \frac{a^2}{4} \times \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{6}} = \frac{a^3}{12\sqrt{2}}
\]

\[
V' = \frac{a^3}{12\sqrt{2}} \quad \ldots \ldots \ldots \ldots (\Pi)
\]

Normal distance \(H_T\) of equilateral triangular faces from the centre of truncated hexahedron: The normal distance \(H_T\) of each of the equilateral triangular faces from the centre C of truncated hexahedron is given as
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It's clear that all 8 congruent equilateral triangular faces are at an equal normal distance $H_T$ from the centre of any truncated hexahedron.

Solid angle ($\omega_T$) subtended by each of the equilateral triangular faces at the centre of truncated hexahedron: we know that the solid angle ($\omega$) subtended by any regular polygon with each side of length $a$ at any point lying at a distance $H$ on the vertical axis passing through the centre of plane is given by “HCR's Theory of Polygon” as follows

$$\omega = 2\pi - 2\pi \sin^{-1}\left(\frac{2H \sin \frac{\pi}{n}}{\sqrt{4H^2 + a^2 \cot^2 \frac{\pi}{n}}}\right)$$

Hence, by substituting the corresponding values in the above expression, we get

$$\omega_T = 2\pi - 2 \times 3 \sin^{-1}\left(\frac{\left(\frac{3 + 2\sqrt{2}}{2}\right) \sin \frac{\pi}{3}}{\sqrt{\left(\frac{3 + 2\sqrt{2}}{2}\right)^2 + a^2 \cot^2 \frac{\pi}{3}}}\right)$$

$$= 2\pi - 6 \sin^{-1}\left(\frac{\left(3 + 2\sqrt{2}\right) \times \frac{\sqrt{3}}{2}}{\sqrt{9 + 8 + 12\sqrt{2} + \frac{1}{3}}}\right) = 2\pi - 6 \sin^{-1}\left(\frac{\left(3 + 2\sqrt{2}\right)\sqrt{3}}{2\sqrt{17 + 12\sqrt{2} + 1}}\right)$$

$$= 2\pi - 6 \sin^{-1}\left(\frac{\sqrt{3}(3 + 2\sqrt{2})}{2\sqrt{6(3 + 2\sqrt{2})}}\right) = 2\pi - 6 \sin^{-1}\left(\frac{\sqrt{3} + 2\sqrt{2}}{2\sqrt{2}}\right) = 2\pi - 6 \sin^{-1}\left(\frac{1}{2} \frac{3 + 2\sqrt{2}}{2}\right)$$

$$\omega_T = 2\pi - 6 \sin^{-1}\left(\frac{1}{2} \frac{3 + 2\sqrt{2}}{2}\right) \approx 0.146577792 \text{ sr} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots 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Mathematical analysis of truncated hexahedron (cube)

Application of HCR’s formula for regular polyhedrons (all five platonic solids)

It’s clear that all 6 congruent regular octagonal faces are at an equal normal distance $H_o$ from the centre of any truncated hexahedron.

It’s also clear from eq(III) & (V) $H_T > H_o$ i.e. the normal distance ($H_T$) of equilateral triangular faces is greater than the normal distance ($H_o$) of regular octagonal faces from the centre of truncated hexahedron i.e. octagonal faces are much closer to the centre as compared to the triangular faces in any truncated hexahedron.

Solid angle ($\omega_o$) subtended by each of the regular octagonal faces at the centre of truncated hexahedron: we know that the solid angle ($\omega$) subtended by any regular polygon is given by “HCR’s Theory of Polygon” as follows

$$\omega = 2\pi - 2\pi \sin^{-1}\left(\frac{2H\sin\frac{\pi}{n}}{4H^2 + a^2\cot^2\frac{\pi}{n}}\right)$$

Hence, by substituting the corresponding values in the above expression, we get

$$\omega_o = 2\pi - 16\sin^{-1}\left(\frac{\left(1 + \sqrt{2}\right)\sqrt{2 - \sqrt{2}}}{2\left(1 + 2 + 2\sqrt{2}\right) + \left(\frac{2 + \sqrt{2}}{2 - \sqrt{2}}\right)^2}\right)$$

$$\omega_o = 2\pi - 16\sin^{-1}\left(\frac{\left(1 + \sqrt{2}\right)\sqrt{2 - \sqrt{2}}}{2\left(\sqrt{2} + \sqrt{2}\right) + \left(\frac{2 + \sqrt{2}}{2 - \sqrt{2}}\right)^2}\right)$$

$$\omega_o = 2\pi - 16\sin^{-1}\left(\frac{\left(1 + \sqrt{2}\right)\sqrt{2 - \sqrt{2}}}{2\sqrt{2} + \sqrt{2}}\right)$$

$$\omega_o = 2\pi - 16\sin^{-1}\left(\frac{\sqrt{2}}{2\sqrt{2} + \sqrt{2}}\right)$$

$$\omega_o = 2\pi - 16\sin^{-1}\left(\frac{1}{2\sqrt{2} + \sqrt{2}}\right) \approx 1.898958046 \text{ sr} \quad \text{.......................... (VI)}$$

Applications of “HCR’s Theory of Polygon” proposed by Mr H.C. Rajpoot (year-2014)
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It’s clear that the solid angle subtended by each of the regular octagonal faces is greater than the solid angle subtended by each of the equilateral triangular faces at the centre of any truncated hexahedron.

Important parameters of a truncated hexahedron:

1. **Inner (inscribed) radius** ($R_i$): It is the radius of the largest sphere inscribed (trapped inside) by a truncated hexahedron. The largest inscribed sphere always touches all 6 congruent regular octagonal faces but does not touch any of 8 equilateral triangular faces at all since all 6 octagonal faces are closer to the centre as compared to all 8 triangular faces. Thus, inner radius is always equal to the normal distance ($H_o$) of octagonal faces from the centre & is given from the eq(V) as follows

$$R_i = H_o = \frac{a}{2} \left(1 + \sqrt{2}\right) \approx 1.207106781a$$

Hence, the volume of inscribed sphere is given as

$$V_{inscribed} = \frac{4}{3} \pi (R_i)^3 = \frac{4}{3} \pi \left(\frac{a}{2} \left(1 + \sqrt{2}\right)\right)^3 \approx 7.367593987a^3$$

2. **Outer (circumscribed) radius** ($R_o$): It is the radius of the smallest sphere circumscribing a given truncated hexahedron or it’s the radius of a spherical surface passing through all 24 vertices of a given truncated hexahedron. It is calculated as follows (See figure 3 above).

$$R_o = \text{distance of any of the vertices from the centre } C = CA = CB = CD$$

In right $\triangle OMA$

$$\sin \angle AOM = \frac{\overline{AO}}{\overline{OA}} \Rightarrow \sin \frac{60^\circ}{2} \overline{OA} = \frac{\overline{AO}}{2} = \frac{\overline{OA}}{2} = \frac{a}{\sqrt{3}} \left(\text{since, } \angle AOB = \frac{2\pi}{3} = 120^\circ\right)$$

In right $\triangle AOC$

$$\overline{CA} = \sqrt{\overline{(OA)}^2 + \overline{(OC)}^2} = \sqrt{\left(\frac{a}{\sqrt{3}}\right)^2 + \left(\frac{3 + 2\sqrt{2}}{2\sqrt{3}}a\right)^2} \left(\text{since, } OC = H_r\right)$$

$$= \frac{\alpha}{\sqrt{3}} + \frac{9 + 8 + 12\sqrt{2}}{12} = \frac{21 + 12\sqrt{2}}{12} = \frac{12}{\sqrt{7} + 4\sqrt{2}} = \frac{21}{\sqrt{7} + 4\sqrt{2}} = \frac{a}{2} \sqrt{7 + 4\sqrt{2}}$$

Hence, the outer radius of truncated hexahedron is given as

$$R_o = \frac{a}{2} \sqrt{7 + 4\sqrt{2}} \approx 1.778823616a$$

Hence, the volume of circumscribed sphere is given as

$$V_{circumscribed} = \frac{4}{3} \pi (R_o)^3 = \frac{4}{3} \pi \left(\frac{a}{2} \sqrt{7 + 4\sqrt{2}}\right)^3 \approx 23.57693199a^3$$

Applications of “HCR’s Theory of Polygon” proposed by Mr H.C. Rajpoot (year-2014)
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3. **Surface area** ($A_s$): We know that a truncated hexahedron has 8 congruent equilateral triangular & 6 regular octagonal faces each of edge length $a$. Hence, its surface area is given as follows

$$A_s = 8 \times (\text{area of equilateral triangle}) + 6 \times (\text{area of regular octagon})$$

We know that area of any regular $n$-polygon with each side of length $a$ is given as

$$A = \frac{1}{4}n a^2 \cot \frac{\pi}{n}$$

Hence, by substituting all the corresponding values in the above expression, we get

$$A_s = 8 \times \left(\frac{1}{4} \times 3 a^2 \cot \frac{\pi}{3}\right) + 6 \times \left(\frac{1}{4} \times 8 a^2 \cot \frac{\pi}{8}\right) = 6a^2 \times \frac{1}{\sqrt{3}} + 12a^2 \cot 22.5^\circ = (2\sqrt{3} + 12\cot 22.5^\circ)a^2$$

$$A_s \approx 32.43466436a^2$$

4. **Volume** ($V$): We know that a truncated hexahedron with edge length $a$ is obtained by truncating a regular hexahedron with edge length $(1 + \sqrt{2})a$ at all its 8 vertices. Thus, 8 congruent right pyramids with equilateral triangular base are truncated off from the parent regular hexahedron. Hence, the volume ($V$) of the truncated hexahedron is given as follows

$$V = (\text{volume of parent regular hexahedron}) - 8 \times (\text{volume of truncated off right pyramid})$$

$$= (1 + 2\sqrt{2} + 3\sqrt{2} + 6)a^3 - 8 \times \left(\frac{a^3}{12\sqrt{2}}\right) \quad (\text{substituting the value of } V' \text{ from eq (II)})$$

$$= (7 + 5\sqrt{2})a^3 - \frac{\sqrt{2}a^3}{3} = \frac{21 + 15\sqrt{2} - \sqrt{2}a^3}{3} = \frac{21 + 14\sqrt{2}a^3}{3} = \frac{7}{3} (3 + 2\sqrt{2})a^3$$

$$V \approx \frac{7}{3} (3 + 2\sqrt{2})a^3 \approx 13.59966329a^3$$

5. **Mean radius** ($R_m$): It is the radius of the sphere having a volume equal to that of a given truncated hexahedron. It is calculated as follows

$$\text{volume of sphere with mean radius } R_m = \text{volume of given truncated hexahedron}$$

$$\frac{4}{3} \pi (R_m)^3 = \frac{7}{3} (3 + 2\sqrt{2})a^3 \Rightarrow (R_m)^3 = \frac{7}{4\pi} (3 + 2\sqrt{2})a^3$$

$$R_m = \left(\frac{7}{4\pi} (3 + 2\sqrt{2})\right)^{\frac{1}{3}} \approx 1.480743548a$$

It’s clear from the above results that $R_i < R_m < R_o$

**Construction of a solid truncated hexahedron**: In order to construct a solid truncated hexahedron with edge length $a$ there are two methods
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1. Construction from elementary right pyramids: In this method, first we construct all elementary right pyramids as follows

Construct 8 congruent right pyramids with equilateral triangular base of side length \( a \) & normal height \( H_T \)
\[
H_T = \frac{(3 + 2\sqrt{2})a}{2\sqrt{3}} \approx 1.602521935a
\]

Construct 6 congruent right pyramids with regular octagonal base of side length \( a \) & normal height \( H_O \)
\[
H_O = \frac{a}{2} \left( 1 + \sqrt{2} \right) \approx 1.207106781a
\]

Now, paste/bond by joining all these right pyramids by overlapping their lateral surfaces & keeping their apex points coincident with each other such that all the edges of each equilateral triangular base (face) coincide with the edges of three octagonal bases (faces). Thus, a solid truncated hexahedron, with 8 congruent equilateral triangular & 6 congruent regular octagonal faces each of edge length \( a \), is obtained.

2. Machining a solid sphere: It is a method of machining, first we select a blank as a solid sphere of certain material (i.e. metal, alloy, composite material etc.) & with suitable diameter in order to obtain the maximum desired edge length of truncated hexahedron. Then, we perform facing operations on the solid sphere to generate 8 congruent equilateral triangular & 6 congruent regular octagonal faces each of equal edge length.

Let there be a blank as a solid sphere with a diameter \( D \). Then the edge length \( a \), of a truncated hexahedron of maximum volume to be produced, can be co-related with the diameter \( D \) by relation of outer radius \( (R_o) \) with edge length \( (a) \) of a truncated hexahedron as follows

\[
R_o = \frac{a}{2} \sqrt{7 + 4\sqrt{2}}
\]

Now, substituting \( R_o = \frac{D}{2} \) in the above expression, we have

\[
\frac{D}{2} = a \sqrt{7 + 4\sqrt{2}} \quad \text{or} \quad D = a \sqrt{7 + 4\sqrt{2}}
\]

\[
a = \frac{D}{\sqrt{7 + 4\sqrt{2}}} \approx 0.281084637D
\]

Above relation is very useful for determining the edge length \( a \) of a truncated hexahedron to be produced from a solid sphere with known diameter \( D \) for manufacturing purposes.

Hence, the maximum volume of truncated hexahedron produced from the solid sphere is given as follows

\[
V_{max} = \frac{7}{3} (3 + 2\sqrt{2})a^3 = \frac{7}{3} (3 + 2\sqrt{2}) \left( \frac{D}{\sqrt{7 + 4\sqrt{2}}} \right)^3 = \frac{7(3 + 2\sqrt{2})D^3}{3(7 + 4\sqrt{2})\sqrt{7 + 4\sqrt{2}}}
\]
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Minimum volume of material removed is given as

\[
(V_{\text{removed}})_{\text{min}} = (\text{volume of solid sphere with diameter } D) - (\text{volume of truncated hexahedron})
\]

\[
= \frac{\pi}{6}D^3 - \frac{7(5 + 2\sqrt{2})D^3}{51\sqrt{7 + 4\sqrt{2}}} = \left(\frac{\pi}{6} - \frac{7(5 + 2\sqrt{2})}{51\sqrt{7 + 4\sqrt{2}}}\right)D^3
\]

\[
(V_{\text{removed}})_{\text{min}} = \left(\frac{\pi}{6} - \frac{7(5 + 2\sqrt{2})}{51\sqrt{7 + 4\sqrt{2}}}\right)D^3 \approx 0.221576143D^3
\]

Percentage (%) of minimum volume of material removed

\[
\% \text{ of } V_{\text{removed}} = \frac{\text{minimum volume removed}}{\text{total volume of sphere}} \times 100
\]

\[
= \left(\frac{\pi}{6} - \frac{7(5 + 2\sqrt{2})}{51\sqrt{7 + 4\sqrt{2}}}\right)D^3 \times 100 - \left(1 - \frac{14(5 + 2\sqrt{2})}{17\pi\sqrt{7 + 4\sqrt{2}}}\right) \times 100 \approx 42.32\
\]

It’s obvious that when a truncated hexahedron of maximum volume is produced from a solid sphere then about 42.32% of material is removed as scraps. Thus, we can select the optimum diameter of blank as a solid sphere to produce a truncated hexahedron of maximum volume (or with maximum desired edge length).

Conclusions: let there be any truncated hexahedron having 8 congruent equilateral triangular & 6 congruent regular octagonal faces each with edge length \(a\) then all its important parameters are calculated/determined as tabulated below

<table>
<thead>
<tr>
<th>Congruent polygonal faces</th>
<th>No. of faces</th>
<th>Normal distance of each face from the centre of the given truncated hexahedron</th>
<th>Solid angle subtended by each face at the centre of the given truncated hexahedron</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equilateral triangle</td>
<td>8</td>
<td>(\frac{(3 + 2\sqrt{2})a}{2\sqrt{3}} \approx 1.682521985a)</td>
<td>(2\pi - 6\sin^{-1}\left(\frac{1}{2}\sqrt{\frac{3 + 2\sqrt{2}}{2}}\right)) \approx 0.146577792 sr</td>
</tr>
<tr>
<td>Regular octagon</td>
<td>6</td>
<td>(\frac{a}{2}(1 + \sqrt{2}) \approx 1.207106781a)</td>
<td>(2\pi - 16\sin^{-1}\left(\frac{1}{2\sqrt{2 + \sqrt{2}}}\right)) \approx 1.898958046 sr</td>
</tr>
</tbody>
</table>
### Mathematical analysis of truncated hexahedron (cube)

**Application of HCR's formula for regular polyhedrons (all five platonic solids)**

<table>
<thead>
<tr>
<th></th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner (inscribed) radius ( R_i )</td>
<td>( R_i = \frac{a}{2} \left( 1 + \sqrt{2} \right) \approx 1.207106781a )</td>
</tr>
<tr>
<td>Outer (circumscribed) radius ( R_c )</td>
<td>( R_c = \frac{a}{2} \sqrt{7 + 4\sqrt{2}} \approx 1.778323646a )</td>
</tr>
<tr>
<td>Mean radius ( R_m )</td>
<td>( R_m = a \left( \frac{7}{4\pi} \left( 3 + 2\sqrt{2} \right) \right)^{\frac{1}{3}} \approx 1.480743548a )</td>
</tr>
<tr>
<td>Surface area ( A_s )</td>
<td>( A_s = \left( 2\sqrt{3} + 12\cot 22.5^\circ \right) a^2 \approx 32.43466436a^2 )</td>
</tr>
<tr>
<td>Volume ( V )</td>
<td>( V = \frac{7}{3} \left( 3 + 2\sqrt{2} \right) a^3 \approx 13.59966329a^3 )</td>
</tr>
</tbody>
</table>

**Note:** Above articles had been developed & illustrated by Mr H.C. Rajpoot (B Tech, Mechanical Engineering)

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**Courtesy:** Advanced Geometry by Harish Chandra Rajpoot